Generalized Phase Crossing Rate and Random FM Noise for Weibull Fading Channels

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***Abstract*— In this paper, exact closed-form expressions are derived for the generalized phase crossing rate as well as for the crossing statistics of random frequency modulation (FM) noise in Weibull fading channels. These expressions specialize to those of the Rayleigh case, which are known in the literature. In addition, the conditional probability density functions of the envelope and the random FM noise conditioned on an arbitrary phase upward crossing level are obtained.**

***Index Terms*— Crossing statistics, phase crossing rate, random FM noise, Weibull fading channels.**

I. INTRODUCTION

The Weibull distribution is an empirical distribution, which was first used as a statistical model for reliability analysis. Its simplicity and flexibility soon paved its way to wireless communications applications. In [1], a physical model for a generalized fading distribution was proposed, in which Weibull appears as a special case. In [2], this model was used in order to obtain higher order statistics for the Weibull environment. In essence, as proposed in [1], the fading model for the Weibull distribution considers a signal composed of a cluster of multipath waves propagating in a non-homogeneous environment. Within this cluster, the phases of the scattered waves are random. The resulting envelope is obtained as a non-linear function of the modulus of the sum of the multipath components. Whereas in [1] the non-linearity is assumed to affect the envelope, in the present work, both envelope and phase are considered to be influenced by it.

In wireless communication systems, the envelope and phase of the received signal vary in a random manner because of multipath fading. It is therefore important to investigate the second order statistics of the fading channels. Besides the statistics of the fading envelope, the statistical properties of the phase process and its derivative are also of interest. For instance, the phase behavior characterization is useful in the design of optimal carrier recovery schemes needed in the synchronization subsystem of coherent receivers [3]. Another notable example involves the performance of FM receivers using a limiter-discriminator for detection, where random FM spikes generated by phase jumps deteriorate the error-rate performance [4]. Thus, the level crossing theory plays a central role in the determination of the statistical properties of the channel phase and random FM noise [5].

In this paper, we derive closed-form expressions for the statistics of random FM noise in Weibull fading channels. Closed-form expressions are also derived for the phase crossing rate (PCR) conditioned on the fact that the fading envelope is within an arbitrary range, whose inferior and superior limits are denoted here as r_1 and r_2, respectively. In addition, the probability density functions of the signal envelope and random FM noise conditioned on the upward crossings of the phase through a fixed level are obtained. The expressions are validated by specializing the general results to some particular cases whose solutions are known.

II. PRELIMINARIES AND THE WEIBULL FADING MODEL

The Rayleigh fading envelope R_l and phase Θ_l are described by

$$R_l = \sqrt{X^2+Y^2} \tag{1a}$$

$$\Theta_l = \arctan\frac{Y}{X} \tag{1b}$$

where X and Y are independent zero-mean Gaussian processes with identical variances σ^2. The time derivative of X and Y are, respectively, denoted as $\dot{X}$ and $\dot{Y}$, which have equal variances expressed as $\dot{\sigma}^2 = 2\pi^2 f_m^2 \sigma^2$, where f_m is the maximum Doppler shift in Hertz [6]. From (1), the Rayleigh complex signal can be expressed as $Z_l = R_l \exp(j\Theta_l)$.

Let R and Θ be random variables representing, respectively, the envelope and phase of the Weibull signal. In the Weibull fading model, the probability density function (PDF) $p_R(r)$ and the cumulative distribution function (CDF) $P_R(r)$ of the envelope are well-known statistics given by

$$p_R(r) = \frac{\alpha r^{\alpha-1}}{\Omega}\exp(-\frac{r^\alpha}{\Omega}) \tag{2a}$$

$$P_R(r) = 1-\exp(-\frac{r^\alpha}{\Omega}) \tag{2b}$$

where α stands for the Weibull fading parameter ($\alpha > 0$) and [1] $\Omega = \mathrm{E}(r^\alpha) = 2\sigma^2$.

According to [1], the resulting complex signal of a Weibull process is a non-linear process obtained not simply as the modulus of the sum of the multipath component, but as this modulus to a certain given exponent. Suppose that such a non-linearity is in the form of the power parameter α so that the resulting complex signal Z is

$$\begin{aligned} Z &= Z_l^{2/\alpha} = R_l^{2/\alpha}\exp(j\,2\,\Theta_l/\alpha) & (3)\\ &= R\exp(j\,\Theta) & (4)\end{aligned}$$

[1] $\mathrm{E}(\cdot)$ denotes the expectation operator

As α increases, the severity of the fading decreases. For the special case of $\alpha = 2$, (3) reduces to the well-known Rayleigh complex process.

Because the non-linearity is in the form of a power, the resulting envelope is observed as the modulus of the multipath Rayleigh component R_l to the power of $2/\alpha > 0$. As seen before, this non-linearity also affects the phase, so that the resulting phase is obtained as a linear function of the phase Rayleigh component Θ_l. Then, from (3) and (4), the relation between the Weibull and the Rayleigh processes is given by

$$R = R_l^{\frac{2}{\alpha}} \tag{5a}$$

$$\Theta = \frac{2\Theta_l}{\alpha} \tag{5b}$$

III. Statistics of Random FM Noise

The random nature of the time-varying phase of the fading signal, denoted as $\dot{\Theta}(t)$, causes a phenomenon known as random FM noise [6]. For the purpose of deriving the statistics of random FM noise and the analysis of the crossing statistics, the joint PDF $p_{R,\dot{R},\Theta,\dot{\Theta}}(r,\dot{r},\theta,\dot{\theta})$ of the processes $R(t)$, $\dot{R}(t)$, $\Theta(t)$, and $\dot{\Theta}(t)$ is required. This joint PDF can be obtained from the joint PDF Rayleigh $p_{R_l,\dot{R}_l,\Theta_1,\dot{\Theta}_l}(r_l,\dot{r}_l,\theta_l,\dot{\theta}_l)$, which is given in [6]. By means of (5) and [6, Eq. 1.3-33], and following the standard statistical procedure of transformation of variates, the Weibull joint PDF can be expressed as

$$p_{R,\dot{R},\Theta,\dot{\Theta}}(r,\dot{r},\theta,\dot{\theta}) = \frac{r^{\alpha}}{4\pi^2\sigma^2\dot{\sigma}^2}\exp\left[-\frac{1}{2}\left(\frac{r^{\alpha}}{\sigma^2} + \frac{\alpha^2 r^{\alpha-2}\dot{r}^2}{4\dot{\sigma}^2} + \frac{r^{\alpha}\alpha^2\dot{\theta}^2}{4\dot{\sigma}^2}\right)\right] \times \frac{1}{|J|} \tag{6}$$

where $|J|$ is the Jacobian of the transformation given by $\frac{16r^{2-\alpha}}{\alpha^4}$. By substituting $|J|$ in (6), the joint PDF of the envelope R and the phase Θ, and their respective time derivatives, can be derived as

$$p_{R,\dot{R},\Theta,\dot{\Theta}}(r,\dot{r},\theta,\dot{\theta}) = \frac{\alpha^4 r^{2\alpha-2}}{64\pi^2\sigma^2\dot{\sigma}^2} \times \exp\left[-\frac{r^{\alpha-2}\left(\alpha^2\sigma^2\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + 4r^2\dot{\sigma}^2\right)}{8\sigma^2\dot{\sigma}^2}\right] \tag{7}$$

where $r \geq 0$, $-\infty < \dot{r} < -\infty$, $-\frac{2\pi}{\alpha} \leq \theta < \frac{2\pi}{\alpha}$, and $-\infty < \dot{\theta} < \infty$. Note that (7) does not depend on the phase θ.

Performing the integrating in (7) with respect to $\dot{r}$, the joint PDF $p_{R,\Theta,\dot{\Theta}}(r,\theta,\dot{\theta})$ can be directly obtained as

$$p_{R,\Theta,\dot{\Theta}}(r,\theta,\dot{\theta}) = \frac{\alpha^3 r^{\frac{3\alpha}{2}-1}}{16\sqrt{2}\pi^{\frac{3}{2}}\sigma^2\dot{\sigma}}\exp\left[-\frac{1}{8}r^{\alpha}\left(\frac{4}{\sigma^2} + \frac{\alpha^2\dot{\theta}^2}{\dot{\sigma}^2}\right)\right] \tag{8}$$

In a similar manner, we found that

$$p_{\Theta,\dot{\Theta}}(\theta,\dot{\theta}) = \frac{\alpha^2}{2\pi\sigma^2\dot{\sigma}}\left(\frac{4}{\sigma^2} + \frac{\alpha^2\dot{\theta}^2}{\dot{\sigma}^2}\right)^{-\frac{3}{2}} \tag{9}$$

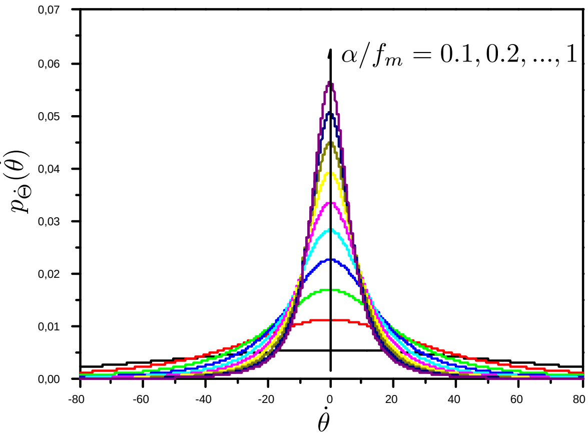

Fig. 1. PDF of $\dot{\Theta}$ for different values of α/f_m

$$p_{\dot{\Theta}}(\dot{\theta}) = \frac{2\alpha}{\sigma^2\dot{\sigma}}\left(\frac{4}{\sigma^2} + \frac{\alpha^2\dot{\theta}^2}{\dot{\sigma}^2}\right)^{-\frac{3}{2}} \tag{10}$$

By setting $\alpha = 2$, (10) reduces to the case Rayleigh fading given in [6, Eq. 1.4-1], as expected. It follows from (10) that the CDF of $\dot{\Theta}(t)$ is obtained in an exact manner as

$$P_{\dot{\Theta}}(\dot{\theta}_0) = \frac{1}{2}\left[1 + \frac{\alpha\dot{\theta}_0}{\dot{\sigma}}\left(\frac{4}{\sigma^2} + \frac{\alpha^2\dot{\theta}_0^{\,2}}{\dot{\sigma}^2}\right)^{-\frac{1}{2}}\right] \tag{11}$$

Again, by setting $\alpha = 2$, (11) reduces to [6, Eq. 1.4-4]. The fact that $p_{\dot{\Theta}}(\dot{\theta})$ is an even function in $\dot{\theta}$ leads to $\mathrm{E}\{\dot{\Theta}(t)\} = 0$ Furthermore, it can be verified that the second moment of $\dot{\Theta}(t)$ equals

$$\mathrm{E}\{\dot{\Theta}(t)^2\} = \int_{-\infty}^{\infty}\dot{\theta}^2 p_{\dot{\Theta}}(\dot{\theta})d\dot{\theta} = \infty \tag{12}$$

Thus, the variance of $\dot{\Theta}(t)$, defined as $\sigma_{\dot{\Theta}}^2 = \mathrm{E}\{\dot{\Theta}(t)^2\} - \mathrm{E}\{\dot{\Theta}(t)^2\}$, is infinite.

For illustration purpose, Figs. 1 and 2 sketch $p_{\dot{\Theta}}(\dot{\theta})$ and $P_{\dot{\Theta}}(\dot{\theta})$ for several values of α/f_m. In Fig. 1, note that the $\dot{\Theta}$ become determinist when α/f_m tends to infinity. From Fig. 2, as α/f_m increases, more quickly the CDF tends towards to the unitary value.

IV. Generalized PCR

The usual (unconditioned) PCR, denoted by $N_{\Theta}(\theta_0)$, is defined as the average number of upward (or downward) crossings per second at a specified phase level θ_0. This definition can be extended to a general case, in which the crossing rate of the phase is conditioned on the arbitrary range r_1 and r_2 of the fading envelope. Thus, a general expression

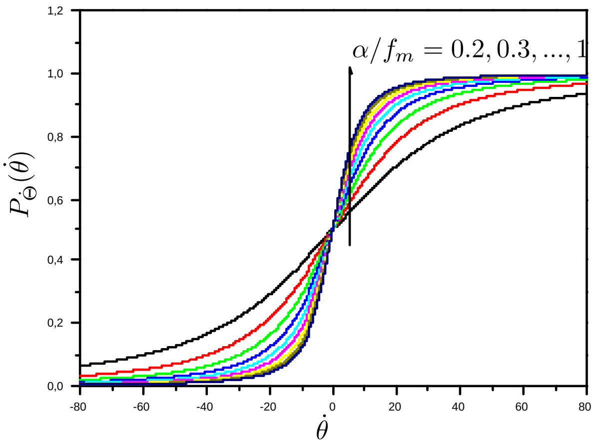

Fig. 2. CDF of $\dot{\Theta}$ for different values of α/f_m

for PCR can be presented as

$$N_{\Theta|R}(\theta_0; r_1, r_2) = \int_0^{\infty} \dot{\theta} p_{\Theta,\dot{\Theta};R}(\theta, \dot{\theta}|r_1 < R < r_2) d\dot{\theta} = \frac{\int_{r_1}^{r_2} \int_0^{\infty} \dot{\theta} p_{R,\Theta,\dot{\Theta}}(r, \theta_0, \dot{\theta}) d\dot{\theta} dr}{P_R(r_2) - P_R(r_1)} \quad (13)$$

Substituting (2b) and (8) into (13), and carrying out some manipulations, the authors have found an exact and closed-form expression for generalized PCR in Weibull fading channels

$$N_{\Theta}(\theta_0; r_1, r_2) = \frac{\dot{\sigma}}{4\pi\sigma} \text{erf}\left(\frac{r_1^{\alpha/2}}{\sqrt{2}\sigma}, \frac{r_2^{\alpha/2}}{\sqrt{2}\sigma}\right) \times \left[\frac{1}{\exp\left(-\frac{r_1^{\alpha}}{2\sigma^2}\right) - \exp\left(-\frac{r_2^{\alpha}}{2\sigma^2}\right)}\right] \quad (14)$$

where $\text{erf}(a, b) = \frac{2}{\sqrt{\pi}} \int_a^b \exp(-t^2) dt$. Note that (14) is independent to the specific phase level θ_0.

In particular, for $\alpha = 2$, we obtain the generalized PCR for the Rayleigh fading channel, which can be expressed as

$$N_{\Theta}(\theta_0; r_1, r_2) = \frac{\dot{\sigma}}{4\pi\sigma} \text{erf}\left(\frac{r_1}{\sqrt{2}\sigma}, \frac{r_2}{\sqrt{2}\sigma}\right) \times \left[\frac{1}{\exp\left(-\frac{r_1^2}{2\sigma^2}\right) - \exp\left(-\frac{r_2^2}{2\sigma^2}\right)}\right] \quad (15)$$

After obtaining the generalized PCR, let us particularized this statistic for the specific case (unconditioned) in which $r_1 = 0$ and $r_2 = \infty$. Then, applying this approach to (14), we obtain

$$N_{\Theta}(\theta_0) = \frac{\dot{\sigma}}{4\pi\sigma} \quad (16)$$

Observe that $N_{\Theta}(\theta_0)$ is independent on the specific phase level θ_0 and the power parameter α. Note also that (16) gives the same result of the Rayleigh case. Although this leads to a same result, (9) is different of the Rayleigh case. Thus, for the unconditional case, it can be observed that the non-linearity, expressed for the α parameter, does not affect the phase process, although the crossing rate of the envelope process [7] be affected. To the best of the authors' knowledge, this has never been reported in the literature.

V. Conditional PDFs of $R(t)$ and $\dot{\Theta}(t)$

This section relates the conditional PDFs of the envelope and the random FM noise $\dot{\Theta}(t)$ conditioned on the upward crossings events occurring at an arbitrary phase crossing [2] level θ_0^+. These conditional PDFs allows us to describe the statistics of $R(t)$ and $\dot{\Theta}(t)$ at the instants when $\Theta(t) = \theta_0^+$. In the following, $p_{R|\theta_0^+}(r_0)$ and $p_{\dot{\Theta}|\theta_0^+}(\dot{\theta}_0)$ denote, respectively, the conditional PDF of $R(t)$ given the phase crossing level θ_0^+ and the conditional PDF of $\dot{\Theta}(t)$ given the phase crossing level θ_0^+.

To obtain the PDFs described above, it is necessary to attain the joint conditional CDF $P_{R,\dot{\Theta}|\theta_0^+}(r_0, \dot{\theta}_0)$ which is defined in [8] and [9] as

$$P_{R,\dot{\Theta}|\theta_0^+}(r_0, \dot{\theta}_0) = \frac{\int_0^{r_0} dr \int_0^{\dot{\theta}_0} \dot{\theta} p_{R,\Theta,\dot{\Theta}}(r, \theta_0^+, \dot{\theta}) d\dot{\theta}}{N_{\Theta}(\theta_0^+)} \quad r_0 \geq 0, \dot{\theta}_0 \geq 0 \quad (17)$$

Replacing (8) and (16) in (17), $P_{R,\dot{\Theta}|\theta_0^+}(r_0, \dot{\theta}_0)$ follows. For the sake of simplicity this expression will be omitted.

From the joint conditional CDF $P_{R,\dot{\Theta}|\theta_0^+}(r_0, \dot{\theta}_0)$, the desired conditional PDF $p_{R|\theta_0^+}(r_0)$ is obtained as

$$p_{R|\theta_0^+}(r_0) = \frac{d}{dr} P_{R,\dot{\Theta}|\theta_0^+}(r_0, \infty) = \frac{\alpha r_0^{\frac{\alpha}{2}-1}}{\sqrt{2\pi}\sigma} \exp\left(-\frac{r_0^{\alpha}}{2\sigma^2}\right) \quad (18)$$

Following a similar rationale, the conditional PDF $p_{\dot{\Theta}|\theta_0^+}(\dot{\theta}_0)$ can be expressed as

$$p_{\dot{\Theta}|\theta_0^+}(\dot{\theta}_0) = \frac{d}{d\dot{\theta}_0} P_{R,\dot{\Theta}|\theta_0^+}(\infty, \dot{\theta}_0) = \frac{2\alpha^2 \dot{\theta}_0}{\sigma^2 \dot{\sigma}} \left(\frac{4}{\sigma^2} + \frac{\alpha^2 \dot{\theta}_0^2}{\dot{\sigma}^2}\right)^{-\frac{3}{2}} \quad (19)$$

Figs. 3 and 4 outline the conditional PDFs for several fading parameters ($\alpha = 1, 2, 3, 4, 5$). In Fig. 3, note that it reduces to a Gaussian function, which is verify in (18) when $\alpha = 2$. Furthermore, observe that R and $\dot{\Theta}$ (presented in Fig. 4) become determinist when α tends to infinity.

[2] The superscript $+$ refer to the upward crossings. Note also that in upward crossings events $0 \leq \dot{\Theta}(t) < \infty$.

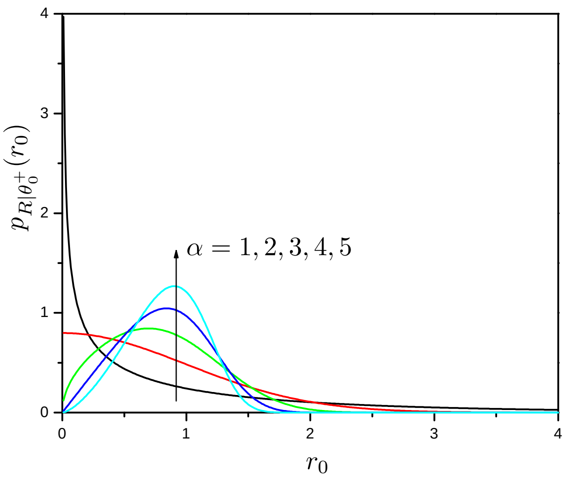

Fig. 3. Conditional PDF $p_{R|\theta_0^+}(r_0)$ for several fading parameters

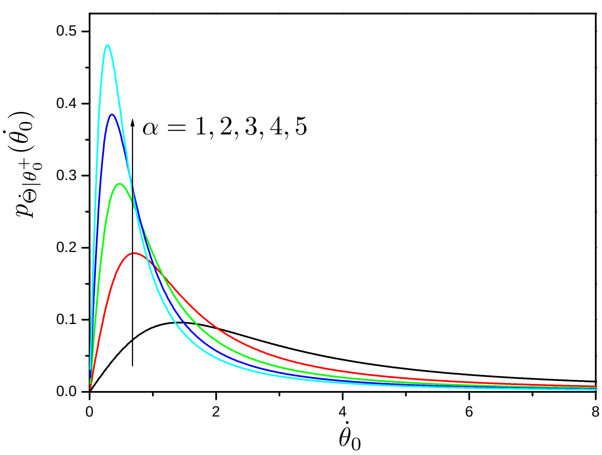

Fig. 4. Conditional PDF $p_{\dot{\Theta}|\theta_0^+}(\dot{\theta}_0)$ for several fading parameters

VI. Conclusions

In this letter, an exact and closed-form expression was derived for the generalized PCR in Weibull fading channels. In addition, statistics of the random FM noise as well as the PDFs of the envelope and random FM noise conditioned at an arbitrary phase level were obtained. These expressions were validated by specializing the general results for the Rayleigh case. It has been found that the unconditioned crossing rate of the phase process for the Weibull fading environment was the same as that of the Rayleigh one, although the joint PDF of the phase and its time derivative were different.

References

[1] M.D.Yacoub. The $\alpha - \mu$ distribution: A general fading distribution. *IEEE Inter. Symp. on Personal, Indoor and Mobile Radio Communications, PIMRC2002*, 2:629–633, Sep. 2002.

[2] M. D. Yacoub, D. B. da Costa, U. S. Dias, and G. Fraindenraich. Joint Statistics for Two Correlated Weibull Variates. *IEEE Ant. and Wireless Prop. Letters*, 4, May 2005.

[3] D.T.Hess. Cycle slipping in a first order phase locked loop. *IEEE Trans. Commun. Technol.*, 35:pp. 255–260, 1968.

[4] R.G.Vaughan. Signals in mobile communications: A review. *IEEE Trans. Vehic. Technol.*, 35:133–145, Nov. 1986.

[5] N. Youssef, W. Elbahri, M. Patzold, and S. Elasmi. On the crossing statistics of phase processes and random fm noise in nakagami-q mobile fading channels. *IEEE Trans. on Wireless Commun.*, 4(1):24–29, Jan. 2005.

[6] W. C. Jakes. *Microwave Mobile Communications*. New York: Wiley, 1974.

[7] N. C. Sagias, D. A.Zogas, G. K. Karagiannidis, and G. S. Tombras. Channel capacity and second-order statistics in weibull fading. 8:377–379, Jun. 2004.

[8] I. B. David and S. Shamai. On the rice model of noise in fm receivers. *IEEE Trans. Commun.*, 34:1406–1419, Nov. 1988.

[9] M. Kac and D. Slepian. Large excursions of gaussian processes. *Ann. Math. statist.*, 30:1215–1228, 1959.